

Lipschitz stability in an inverse problem for the Kuramoto-Sivashinsky equation

Lucie Baudouin*, Eduardo Cerpa†, Emmanuelle Crépeau‡ and Alberto Mercado†

Abstract

This paper presents an inverse problem for the nonlinear 1-d Kuramoto-Sivashinsky (K-S) equation. More precisely, we study the nonlinear inverse problem of retrieving the anti-diffusion coefficient from the measurements of the solution on a part of the boundary and at some positive time everywhere. Uniqueness and Lipschitz stability for this inverse problem are proven with the Bukhgeim-Klibanov method. The proof is based on a global Carleman estimate for the linearized K-S equation.

Keywords: Inverse problem, Kuramoto-Sivashinsky equation, Carleman estimate

AMS subject classifications: 35R30, 35K55.

1 Introduction

This paper focuses on an inverse problem that consists in the determination of a coefficient of a partial differential equation from the partial knowledge of a given single solution of the equation. For the solution of this class of problems (single-measurement inverse problems), the Bukhgeim-Klibanov method was introduced in [6] (see also [12, 13]). This method, which is based on Carleman estimates, allows to prove uniqueness, i.e. that each measurement corresponds to only one coefficient. Regarding the continuity of this inverse problem, the first Lipschitz stability result for a multidimensional wave equation was obtained by Puel and Yamamoto [14] by using a method based on [6]. Since then, this method has been applied to other inverse problems including Lipschitz stability for the Schrödinger equation [1] and Logarithmic stability for the wave equation [2, 3].

*CNRS ; LAAS ; 7 avenue du colonel Roche, F-31077 Toulouse Cedex 4, France

Université de Toulouse ; UPS, INSA, INP, ISAE, UT1, UTM ; LAAS ; F-31077 Toulouse, France.

E-mail: lucie.baudouin@laas.fr

†Departamento de Matemática, Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile.

E-mail: eduardo.cerpa@usm.cl, alberto.mercado@usm.cl

‡Laboratoire de Mathématiques, Université de Versailles Saint-Quentin en Yvelines, 78035 Versailles, France.

E-mail: emmanuelle.crepeau@math.uvsq.fr

This approach was extended to a parabolic equation in [10]. Since then, this type of inverse problems for parabolic equations has received a large amount of attention. The primary difference with respect to hyperbolic inverse problems is that parabolic problems are not time-reversible: therefore, an additional measurement must be added if that method is applied. As one can read in the discussion of the introduction of [10], the knowledge of the full-state of the solution for some positive time is required. To prove the Lipschitz stability without this assumption, which is usually needed when global Carleman inequalities are used, is still an open problem.

Recent results regarding linear parabolic problems can be found in [4] (discontinuous coefficient), [8] (systems), [11] (network) and the references therein. In [5, 9, 15], nonlinear parabolic equations were even considered.

In this paper, the Kuramoto-Sivashinsky (K-S) equation is considered, which is a 1D nonlinear fourth-order parabolic equation. This equation is used to model the physical phenomena of plane flame propagation: it describes the combined influence of diffusion and thermal conduction of gas on the stability of a plane flame front. In this nonlinear partial differential equation, the fourth-order term models the diffusion, and the second-order term models the incipient instabilities. To the knowledge of the authors there are no results in the literature concerning the determination of coefficients for this nonlinear equation. However, a Carleman estimate has been used to obtain the null-controllability of the K-S equation in reference [7] for the constant coefficient case. We consider the inverse problem of retrieving the anti-diffusion coefficient γ from boundary measurements of the solution. Since the linearized equation is parabolic, boundary measurements are not sufficient and we must consider an additional measurement of the full solution for a given time T_0 (as in [4, 10] among others).

The K-S equation with non-constant coefficients describing the diffusion $\sigma = \sigma(x)$, and the anti-diffusion $\gamma = \gamma(x)$, is given as

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + yy_x = g, & \forall (t, x) \in Q, \\ y(t, 0) = h_1(t), \quad y(t, 1) = h_2(t), & \forall t \in (0, T), \\ y_x(t, 0) = h_3(t), \quad y_x(t, 1) = h_4(t), & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, 1), \end{cases} \quad (1)$$

where $Q := (0, T) \times (0, 1)$, $\sigma : [0, 1] \rightarrow \mathbb{R}_+^*$, and the functions y_0, g, h_j are the initial condition, the source term and the boundary data respectively. All of these terms are assumed to be known and compatible.

The first result involves the local well-posedness of the nonlinear equation (1). A less regular framework can be used for this equation. However, the method applied in this paper requires the solution and its time-derivative to be at least in $L^2(0, T; H^4(0, 1))$. Therefore,

let us introduce the following notations for the spaces appearing in this paper:

$$\begin{aligned}\mathcal{Y}_k &:= C([0, T]; H^k(0, 1)) \cap L^2(0, T; H^{k+2}(0, 1)), \quad \text{for } k \in \mathbb{N}; \\ \mathcal{F} &:= \{f \in L^2(0, T; H^4(0, 1)) / f_t \in L^2(0, T; L^2(0, 1))\}; \\ \mathcal{Z} &:= \{z \in \mathcal{Y}_6 / z_t \in \mathcal{Y}_2\}.\end{aligned}\tag{2}$$

Theorem 1.1 *Let $\gamma \in H^4(0, 1)$ and $\sigma \in H^4(0, 1)$ be such that*

$$\forall x \in (0, 1), \sigma(x) \geq \sigma_0 > 0.\tag{3}$$

Let $y_0 \in H^6(0, 1)$, $g \in \mathcal{F}$, and $h_j \in H^2(0, T)$ for $j = 1, \dots, 4$. Assume also that y_0 and $(h_j)_{j=1, \dots, 4}$ satisfy the compatibility conditions $y_0(0) = h_1(0)$, $y_{0,x}(0) = h_3(0)$, $y_0(1) = h_2(0)$, $y_{0,x}(1) = h_4(0)$.

Therefore, there exists $\varepsilon > 0$ such that if

$$\|y_0\|_{H^6(0,1)} \leq \varepsilon, \quad \|g\|_{\mathcal{F}} \leq \varepsilon, \quad \|h_j\|_{H^2(0,T)} \leq \varepsilon \text{ for } j = 1, \dots, 4,\tag{4}$$

then the K-S equation (1) has a unique solution $y \in \mathcal{Z}$.

Once the existence of solutions to the K-S equation has been established (see Section 2), the following inverse problem is addressed:

Is it possible to retrieve the anti-diffusion coefficient $\gamma = \gamma(x)$ from the measurement of $y_{xx}(t, 0)$ and $y_{xxx}(t, 0)$ on $(0, T)$ and from the measurement of $y(T_0, x)$ on $(0, 1)$, where y is the solution to Equation (1) and $T_0 \in (0, T)$?

A local answer for this nonlinear inverse problem is given (see section 4). To be more specific, let $\tilde{\gamma}$ be fixed. We denote by \tilde{y} the solution to Equation (1) with γ replaced by $\tilde{\gamma}$. This paper focuses on the following two questions.

Uniqueness: Do the equalities of the measurements $\tilde{y}_{xx}(t, 0) = y_{xx}(t, 0)$ and $\tilde{y}_{xxx}(t, 0) = y_{xxx}(t, 0)$ for $t \in (0, T)$ and $\tilde{y}(T_0, x) = y(T_0, x)$ for $x \in (0, 1)$ imply $\tilde{\gamma} = \gamma$ on $(0, 1)$?

Stability: Is it possible to estimate $\|\tilde{\gamma} - \gamma\|_{L^2(0,1)}$ by suitable norms $\|\tilde{y}(T_0, x) - y(T_0, x)\|$ in space and $\|\tilde{y}_{xx}(t, 0) - y_{xx}(t, 0)\|$, $\|\tilde{y}_{xxx}(t, 0) - y_{xxx}(t, 0)\|$ in time?

To answer these questions, we use the Bukhgeim-Klibanov method. First, a global Carleman estimate for the linearized K-S equation with non-constant coefficients is obtained. It is then used to prove the primary result which can be stated as follows.

Theorem 1.2 *Let us consider $\sigma \in H^4(0, 1)$, $\gamma \in H^4(0, 1)$ and compatible data y_0 , g and h_j regular enough such that the solutions of (1) belong to $H^1(0, T; H^4(0, 1))$.*

Let us denote y the solution of equation (1) associated to γ and \tilde{y} the solution associated to a fixed coefficient $\tilde{\gamma} \in H^4(0, L)$. We assume that there exists $r > 0$ and $T_0 \in (0, T)$ such that

$$\inf \{|\tilde{y}_{xx}(T_0, x)|, x \in (0, 1)\} \geq r,\tag{5}$$

Let $\mathcal{U} = \{\gamma \in H^4(0,1) / \|\gamma\|_{L^\infty(0,1)} \leq M_1, \|y\|_{H^1(0,T;H^4(0,1))} \leq M_2\}$. Then there exists a positive constant C depending on the parameters (T, M_1, M_2, r) , such that for every $\gamma \in \mathcal{U}$,

$$\begin{aligned} \frac{1}{C} \|\gamma - \tilde{\gamma}\|_{L^2(0,1)}^2 &\leq \|y_{xx}(\cdot, 0) - \tilde{y}_{xx}(\cdot, 0)\|_{H^1(0,T)}^2 + \|y_{xxx}(\cdot, 0) - \tilde{y}_{xxx}(\cdot, 0)\|_{H^1(0,T)}^2 \\ &\quad + \|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^4(0,1)}^2 + \|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^1(0,1)}^4 \\ &\leq C \left(\|y - \tilde{y}\|_{H^1(0,T;H^4(0,1))}^2 + \|y - \tilde{y}\|_{L^\infty(0,T;H^1(0,1))}^4 \right). \end{aligned} \quad (6)$$

This two-sided inequality gathers two complementary informations, namely, the stability of the inverse problem (with the first estimate) and the regularity of the measurements. Indeed, the second estimate in (6) indicates that the required measurements are finite if y and \tilde{y} belong to the space $H^1(0,T;H^4(0,1))$ and this is true if y and \tilde{y} are solutions provided by Theorem 1.1 under the hypothesis (3) and (4).

Remark 1.3 *As stated above, an internal measurement at $t = T_0$ is required if this method is used to solve this type of problem for parabolic equations. Nevertheless, this is probably a technical point and there is no counter-example that demonstrates whether this assumption is required. In [15], uniqueness (but not stability) is proven using a very different technique in an inverse problem for a parabolic equation and without any internal measurements in the entire space domain.*

Remark 1.4 *In this paper, the boundary measurements are located at $x = 0$, but the result would be the same if we measure at $x = 1$ instead. Indeed, the choice of a suitable weight function in the proof of the Carleman estimate in Section 3 is critical to impose the side of measurement.*

This article is organized as follows. The well-posedness result stated in Theorem 1.1 is proved in Section 2. A global Carleman estimate for a general K-S equation is given and proved in Section 3. Finally, Section 4 contains the use of the Bukhgeim-Klibanov method to prove the Lipschitz stability of the inverse problem stated in Theorem 1.2.

2 On the Cauchy problem for KS equation

This section presents a proof of Theorem 1.1 in a more general case including time dependent lower-order coefficients. We consider the following K-S system

$$\begin{cases} y_t + (\sigma(x)y_{xx})_{xx} + \gamma(x)y_{xx} + G_1y_x + G_2y + yy_x = g, & \forall (t, x) \in Q, \\ y(t, 0) = h_1(t), \quad y(t, 1) = h_2(t), & \forall t \in (0, T), \\ y_x(t, 0) = h_3(t), \quad y_x(t, 1) = h_4(t), & \forall t \in (0, T), \\ y(0, x) = y_0(x), & \forall x \in (0, 1), \end{cases} \quad (7)$$

where G_1, G_2 belongs to $H^1(0,T;H^4(0,1))$, $g \in \mathcal{F}$ and $y_0 \in H^6(0,1)$ is compatible with $h_j \in H^2(0,T)$ for $j = 1, \dots, 4$. Recall that the coefficients satisfy $\gamma \in H^4(0,1)$, $\sigma \in H^4(0,1)$

and hypothesis (3).

First, the main part of the linear differential operator is utilized in the next proposition.

Proposition 2.1 *Let $z_0 \in H^6 \cap H_0^2(0, 1)$ and $f \in \mathcal{F}$. Then, the following equation*

$$\begin{cases} z_t + (\sigma(x)z_{xx})_{xx} = f, & \forall (t, x) \in Q, \\ z(t, 0) = 0, \quad z(t, 1) = 0, & \forall t \in (0, T), \\ z_x(t, 0) = 0, \quad z_x(t, 1) = 0, & \forall t \in (0, T), \\ z(0, x) = z_0(x), & \forall x \in (0, 1), \end{cases} \quad (8)$$

has a unique solution $z \in \mathcal{Z}$ and there exists $C > 0$ such that

$$\|z\|_{\mathcal{Z}} \leq C (\|f\|_{\mathcal{F}} + \|z_0\|_{H^6}).$$

Proof. The operator

$$\begin{aligned} H^4 \cap H_0^2(0, 1) \cap L^2(0, 1) &\longrightarrow L^2(0, 1) \\ w &\longmapsto (\sigma(x)w''(x))'', \end{aligned}$$

is simultaneously positive, coercive and self-adjoint. Moreover, its inverse is compact: thus, it generates a strongly continuous semigroup in $L^2(0, 1)$. Therefore, for each $z_0 \in H^4 \cap H_0^2(0, 1)$ and $f \in C^1([0, T]; L^2(0, 1))$, Equation (8) has a unique solution $z \in C([0, T]; H^4 \cap H_0^2(0, 1)) \cap C^1([0, T]; L^2(0, 1))$.

We will demonstrate that the solutions $z \in \mathcal{Z}$ (refer to the notation introduced in (2)), can be obtained by taking z_0 and f sufficiently regular.

We now search for some energy estimates that indicate the space that the solutions lie on depending on the regularity of the data. Suppose that there are solutions sufficiently regular to perform the following computations. Equation (8) is multiplied by z and integrated over $(0, 1)$. Some integrations by parts give

$$\frac{d}{dt} \left(\int_0^1 |z|^2 \right) + \int_0^1 |z_{xx}|^2 \leq C \left(\int_0^1 |f|^2 + \int_0^1 |z|^2 \right). \quad (9)$$

Throughout this paper, C denotes a positive constant that varies from line to line. Using Gronwall's lemma, we obtain

$$\int_0^1 |z|^2 \leq C \left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (10)$$

Then, (9) is integrated over $[0, T]$ and (10) is used to get

$$\iint_Q |z_{xx}|^2 \leq C \left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (11)$$

Inequalities (10) and (11) finally imply that

$$\|z\|_{\mathcal{Y}_0}^2 \leq C \left(\iint_Q |f|^2 + \int_0^1 |z_0|^2 \right). \quad (12)$$

Equation (8) is multiplied by $(\sigma z_{xx})_{xx}$ and integrated over $(0, 1)$. Some integrations by parts give also

$$\frac{1}{2} \frac{d}{dt} \left(\int_0^1 \sigma |z_{xx}|^2 \right) + \int_0^1 |(\sigma z_{xx})_{xx}|^2 = \int_0^1 f (\sigma z_{xx})_{xx}.$$

We can then write

$$\frac{d}{dt} \left(\int_0^1 \sigma |z_{xx}|^2 \right) + \int_0^1 \sigma |(z_{xx})_{xx}|^2 \leq \int_0^1 |f|^2.$$

Thus, we obtain

$$\|z\|_{\mathcal{Y}_2}^2 \leq C \left(\int_Q |f|^2 + \int_0^1 |z_0''|^2 \right). \quad (13)$$

On the other hand, Equation (8) is derived with respect to time. Thus $q := z_t$ satisfies

$$\begin{cases} q_t + (\sigma(x)q_{xx})_{xx} = f_t, & \forall (t, x) \in Q, \\ q(t, 0) = 0, \quad q(t, 1) = 0, & \forall t \in (0, T), \\ q_x(t, 0) = 0, \quad q_x(t, 1) = 0, & \forall t \in (0, T), \\ q(0, x) = f(0, x) - (\sigma z_0''(x))'', & \forall x \in (0, 1). \end{cases} \quad (14)$$

Using estimate (13), we obtain $q \in \mathcal{Y}_2$ if $(f(0, x) - (\sigma z_0''(x))'') \in H^2(0, 1)$ and $f_t \in L^2(0, T; L^2(0, 1))$. These hypotheses are fulfilled if $z_0 \in H^6 \cap H_0^2(0, 1)$ and $f \in \mathcal{F}$. Note that $\mathcal{F} \subset C([0, T]; H^2(0, 1))$. From the equation satisfied by z and the fact that $f \in \mathcal{F}$ and $z_t \in \mathcal{Y}_2$, we determine that $z \in \mathcal{Y}_6$, which concludes the proof of Proposition 2.1. \square

Then, we focus on the linear problem with non-homogenous boundary conditions and low-order coefficients that depend on time.

Proposition 2.2 *Let $z_0 \in H^6(0, 1)$, $\hat{f} \in \mathcal{F}$, $G_1, G_2 \in H^1(0, T; H^4(0, 1))$ and $h_j \in H^2(0, T)$ for $j = 1, \dots, 4$ satisfying the compatibility conditions with z_0 . Then, the equation*

$$\begin{cases} z_t + (\sigma(x)z_{xx})_{xx} + \gamma(x)z_{xx} + G_1 z_x + G_2 z = \hat{f}, & \forall (t, x) \in Q, \\ z(t, 0) = h_1(t), \quad z(t, 1) = h_2(t), & \forall t \in (0, T), \\ z_x(t, 0) = h_3(t), \quad z_x(t, 1) = h_4(t), & \forall t \in (0, T), \\ z(0, x) = z_0(x), & \forall x \in (0, 1), \end{cases} \quad (15)$$

has a unique solution $z \in \mathcal{Z}$ and there exists $C > 0$ such that

$$\|z\|_{\mathcal{Z}} \leq C \left(\|\hat{f}\|_{\mathcal{F}} + \|z_0\|_{H^6} + \sum_{j=1}^4 \|h_j\|_{H^2} \right).$$

Proof. We first prove this result for null boundary data (i.e. for $h_j = 0$ for $j = 1, \dots, 4$ and therefore $z_0 \in H^6 \cap H_0^2(0, 1)$).

For any $\hat{w} \in \mathcal{Z}$, $\Pi(\hat{w})$ is defined as the solution of (8) with $f = (\hat{f} - \gamma(x)\hat{w}_{xx} - G_1\hat{w}_x - G_2\hat{w})$. Note that $f \in \mathcal{F}$ and therefore $\Pi(\hat{w}) \in \mathcal{Z}$ is well defined.

If T is small enough, then Π is a contraction. Indeed, for any $w, \hat{w} \in \mathcal{Z}$, we have (the space $L^m(0, T; H^n(0, 1))$ is denoted as $L^m(H^n)$)

$$\begin{aligned} \|\Pi(\hat{w}) - \Pi(w)\|_{\mathcal{Z}} &\leq C \|\gamma(x)(w_{xx} - \hat{w}_{xx}) + G_1(w_x - \hat{w}_x) + G_2(w - \hat{w})\|_{\mathcal{F}} \\ &\leq C \|w - \hat{w}\|_{L^2(H^6)} + C \|w_t - \hat{w}_t\|_{L^2(H^2)} \end{aligned} \quad (16)$$

$$\begin{aligned} &\leq CT^{\frac{1}{4}} \|w - \hat{w}\|_{L^4(H^6)} + CT^{\frac{1}{4}} \|w_t - \hat{w}_t\|_{L^4(H^2)} \\ &\leq CT^{\frac{1}{4}} \|w - \hat{w}\|_{\mathcal{Y}_6} + CT^{\frac{1}{4}} \|w_t - \hat{w}_t\|_{\mathcal{Y}_2} \\ &\leq CT^{\frac{1}{4}} \|w - \hat{w}\|_{\mathcal{Z}}. \end{aligned} \quad (17)$$

Hence, the operator Π has a unique fixed point in \mathcal{Z} , which is the solution of (15) with $h_j = 0$ for $j = 1, \dots, 4$. Using standard arguments and the linearity of this equation, the solution can be extended to a larger time interval.

In order to prove the general case, take $h_j \in H^2(0, T)$, $j = 1, \dots, 4$ compatible with z_0 . It is not difficult to find a function $\psi \in H^2(0, T; C^\infty([0, 1]))$ satisfying the boundary conditions of (15). For instance take $\psi(x, t) = \sum_{j=1}^4 p_j(x) h_j(t)$ where $p_1(x) = 2x^3 - 3x^2 + 1$, $p_2(x) = -2x^3 + 3x^2$, $p_3(x) = x^3 - 2x^2 + x$ and $p_4(x) = x^3 - x^2$. In particular we have $L\psi := \psi_t + (\sigma(x)\psi_{xx})_{xx} + \gamma(x)\psi_{xx} + G_1\psi_x + G_2\psi \in \mathcal{F}$. Then, if w is the solution of equation (15) with null boundary data, initial condition $w_0 - \psi(\cdot, 0)$, and right-hand side equal to $\hat{f} - L\psi$, let us define $z = w + \psi$. It is not difficult to see that z is the required solution. \square

Remark 2.3 *The third-order term z_{xxx} can be added to Equation (15). Indeed, in that case (16) becomes $C\|w - \hat{w}\|_{L^2(H^7)} + C\|w_t - \hat{w}_t\|_{L^2(H^3)}$, which is bounded by*

$$CT^{\frac{1}{4}}\|w - \hat{w}\|_{L^\infty(H^6)}^{1/2}\|w - \hat{w}\|_{L^2(H^8)}^{1/2} + CT^{\frac{1}{4}}\|w_t - \hat{w}_t\|_{L^\infty(H^2)}^{1/2}\|w_t - \hat{w}_t\|_{L^2(H^4)}^{1/2}.$$

This last expression is bounded by (17). The remainder of the proof is the same.

Again, by using a fixed point theorem, we can prove Theorem 1.1 for equation (7). Let $y_0 \in H^6(0, 1)$, $h_j \in H^2(0, 1)$ compatible with y_0 , and $g \in \mathcal{F}$. For any $v \in \mathcal{Z}$, we define $\Lambda(v)$ as the solution of (15) with $\hat{f} = (g - vv_x)$ and $z_0 = y_0$. Note that $\hat{f} \in \mathcal{F}$ and therefore $\Lambda(v) \in \mathcal{Z}$ is well defined. Indeed, if $v \in \mathcal{Y}_3$ and $v_t \in \mathcal{Y}_0$, then we have

$$(vv_x)_{xxxx} = (10v_{xx}v_{xxx} + 5v_xv_{xxxx} + vv_{xxxx}) \in L^2(0, T; L^2(0, 1))$$

and

$$(vv_x)_t = v_tv_x + vv_{xt} \in L^2(0, T; L^2(0, 1)).$$

Furthermore, we can prove

$$\begin{aligned} \|\Lambda(v)\|_{\mathcal{Z}} &\leq C \left\{ \|g\|_{\mathcal{F}} + \|vv_x\|_{\mathcal{F}} + \|y_0\|_{H^6} + \sum_{j=1}^4 \|h_j\|_{H^2} \right\} \\ &\leq C \left\{ \|g\|_{\mathcal{F}} + \|v\|_{\mathcal{Z}}^2 + \|y_0\|_{H^6} + \sum_{j=1}^4 \|h_j\|_{H^2} \right\}. \end{aligned} \quad (18)$$

Let $\varepsilon > 0$ to be chosen later and suppose that y_0 , h_j and g satisfy (4). Consider v such that $\|v\|_{\mathcal{Z}} \leq r$ with $r > 0$ satisfying $C(6\varepsilon + r^2) < r$. From (18), we obtain $\|\Lambda(v)\|_{\mathcal{Z}} < r$. Thus, the application Λ maps the ball $B_r := \{v \in \mathcal{Z} / \|v\|_{\mathcal{Z}} \leq r\}$ into itself.

We will now prove that $\Lambda : B_r \rightarrow B_r$ is a contraction. For any $z, v \in B_r$, $\Lambda(z) - \Lambda(v)$ is the solution of (15) with $z_0 = 0$, $h_j = 0$ for $j = 1, \dots, 4$ and $\hat{f} = vv_x - zz_x$. We obtain the estimate

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \leq C\|vv_x - zz_x\|_{\mathcal{F}} \leq C\|(v - z)v_x\|_{\mathcal{F}} + \|z(v_x - z_x)\|_{\mathcal{F}}.$$

Using the definition of the space \mathcal{F} , $v, z \in C([0, 1]; H^6(0, 1)) \hookrightarrow L^\infty(0, T; W^{5,\infty}(0, 1))$ and $v_t, z_t \in C([0, 1]; H^6(0, 1)) \hookrightarrow L^\infty(0, T; W^{1,\infty}(0, 1))$, we obtain

$$\|\Lambda(z) - \Lambda(v)\|_{\mathcal{Z}} \leq C(\|v\|_{\mathcal{Z}} + \|z\|_{\mathcal{Z}})\|v - z\|_{\mathcal{Z}} \leq 2Cr\|v - z\|_{\mathcal{Z}},$$

which implies that Λ is a contraction if r is chosen small enough. More precisely, we can choose r, ε such that $2Cr < 1$ and $C(2\varepsilon + r^2) < r$. Hence, the map Λ has a unique fixed point $y \in \mathcal{Z}$, which is the unique solution of (7).

Thus, we have proven Theorem 1.1.

3 Global Carleman inequality

In this section, a global Carleman inequality will be proven for the linearized K-S equation.

We define the space

$$\mathcal{V} = \{v \in L^2(0, T; H^4 \cap H_0^2(0, 1)) \mid Lv \in L^2((0, T) \times (0, 1))\} \quad (19)$$

where

$$Lv = v_t + (\sigma v_{xx})_{xx} + q_2 v_{xx} + q_1 v_x + q_0 v$$

with $q_j \in L^\infty(\Omega)$ for $j = 0, 1, 2$.

Consider $\beta \in C^4([0, 1])$ such that for some $r > 0$ we have, for all $x \in (0, 1)$:

$$0 < r \leq \frac{d^k \beta}{dx^k}(x), \quad \text{for } k = 0, 1, \quad (20)$$

$$\frac{d^2 \beta}{dx^2}(x) \leq -r < 0, \quad (21)$$

$$|\sigma_x(x) \beta_x(x)| \leq \frac{r}{4} \min_{z \in [0, 1]} \{\sigma(z)\}. \quad (22)$$

For instance, if σ is constant, we can consider $\beta(x) = \sqrt{1+x}$.

On the other hand, given $T_0 \in (0, T)$ we can choose $\phi_0 \in C^1([0, T])$ such that

$$\phi_0(0) = \phi_0(T) = 0, \quad \text{and} \quad (23)$$

$$0 < \phi_0(t) \leq \phi_0(T_0) \quad \text{for each } t \in (0, T). \quad (24)$$

For example, if $T_0 = T/2$, we can use $\phi_0(t) = t(T-t)$.

We finally define the function

$$\phi(t, x) = \frac{\beta(x)}{\phi_0(t)}, \quad (25)$$

for $(t, x) \in (0, T) \times [0, 1]$, which is the weight function of the Carleman estimate and has a crucial role in the following result.

Theorem 3.1 *Let ϕ be a function as (25) and $m > 0$. Then there exists $\lambda_0 > 0$ and a constant $C = C(T, \lambda_0, r, m) > 0$ such that if $\|q_i\|_{L^\infty((0, T) \times (0, 1))} \leq m$ for $i = 0, 1, 2$ then we have*

$$\begin{aligned} & \int_0^T \int_0^1 e^{-2\lambda\phi} \left(\frac{|v_t|^2 + |(\sigma v_{xx})_{xx}|^2}{\lambda\phi} + \lambda^7 \phi^7 |v|^2 + \lambda^5 \phi^5 |v_x|^2 + \lambda^3 \phi^3 |v_{xx}|^2 + \lambda\phi |v_{xxx}|^2 \right) dx dt \\ & \leq C \int_0^T \int_0^1 e^{-2\lambda\phi} |Lv|^2 dx dt \\ & + C \int_0^T e^{-2\lambda\phi(t, 0)} \left(\lambda^3 \phi_x^3(t, 0) \sigma(0)^2 |v_{xx}(t, 0)|^2 + \lambda\phi_x(t, 0) \sigma^2(0) |v_{xxx}(t, 0)|^2 \right) dt \end{aligned} \quad (26)$$

for all $v \in \mathcal{V}$, for all $\lambda \geq \lambda_0$.

Proof. Consider the following operator P defined in $\mathcal{W}_\lambda := \{e^{-\lambda\phi}v : v \in \mathcal{V}\}$ by

$$Pw = e^{-\lambda\phi}L(e^{\lambda\phi}w).$$

We then obtain the decomposition $Pw = P_1w + P_2w + Rw$, where

$$P_1w = 6\lambda^2\phi_x^2\sigma w_{xx} + \lambda^4\phi_x^4\sigma w + (\sigma w_{xx})_{xx} + 6\lambda^2(\phi_x^2\sigma)_x w_x \quad (27)$$

$$P_2w = w_t + 4\lambda^3\phi_x^3\sigma w_x + 4\lambda\phi_x\sigma w_{xxx} + 4\lambda^3\phi_x(\phi_x^2\sigma)_x w \quad (28)$$

$$\begin{aligned} Rw = & \lambda\phi_t w + 2\lambda\phi_x\sigma_{xx}w_x + \lambda^2\phi_x^2\sigma_{xx}w + \lambda\phi_{xx}\sigma_{xx}w \\ & + 6\lambda\phi_x\sigma_x w_{xx} + 6\lambda^2\phi_x\phi_{xx}\sigma_x w + 6\lambda\phi_{xx}\sigma_x w + 2\lambda\phi_{xxx}\sigma_x w \\ & + 4\lambda^2\phi_x\phi_{xxx}\sigma w + 6\lambda\phi_{xx}\sigma w_{xx} + 3\lambda^2\phi_{xx}^2\sigma w + 4\lambda\phi_{xxx}\sigma w_x \\ & + \lambda\phi_{xxx}\sigma w + q_0 w + q_1 w_x + q_1 \lambda\phi_x w \\ & + q_2 w_{xx} + 2\lambda q_2 \phi_x w_x + \lambda^2 q_2 \phi_x^2 w + \lambda\phi_{xx} q_2 w \\ & - 2\lambda^3 \phi_x^2 \phi_{xxx} \sigma w - 2\lambda^3 \phi_x^3 \sigma_x w. \end{aligned} \quad (29)$$

Thus,

$$\|Pw - Rw\|_{L^2(Q)}^2 = \|P_1w\|_{L^2(Q)}^2 + 2\langle P_1w, P_2w \rangle + \|P_2w\|_{L^2(Q)}^2$$

where $\langle \cdot, \cdot \rangle$ is the $L^2(Q)$ scalar product.

For any $v \in \mathcal{V}$ we obtain $v_t \in L^2(0, T; L^2(0, 1))$ and then $v \in C([0, T]; L^2(0, 1))$. From the construction of ϕ (see (23)), we obtain $w \in C([0, T]; L^2(0, 1))$ and $w(x, 0) = w(x, T) = 0$ for any $w \in \mathcal{W}_\lambda$.

Let us define the notations

$$I(w) = -6\lambda^7 \iint_Q \phi_x^6 \phi_{xx} \sigma^2 |w|^2,$$

$$I(w_x) = -\lambda^5 \iint_Q \phi_x^4 \sigma (30\phi_{xx}\sigma + 12\phi_x\sigma_x) |w_x|^2,$$

$$I(w_{2x}) = -\lambda^3 \iint_Q \phi_x^2 \sigma (58\phi_{xx}\sigma + 40\phi_x\sigma_x) |w_{xx}|^2,$$

$$I(w_{3x}) = -\lambda \iint_Q \sigma (2\phi_{xx}\sigma - 4\phi_x\sigma_x) |w_{xxx}|^2,$$

and

$$I_x = \int_0^T (10\lambda^3 \phi_x^3 \sigma^2 |w_{xx}|^2 + 2\lambda \phi_x \sigma \sigma_{xx} |w_{xx}|^2 + 2\lambda \phi_x \sigma^2 |w_{xxx}|^2) \Big|_{x=0}^1 dx dt.$$

The following weighted norm is defined, for any $w \in \mathcal{W}_\lambda$, as

$$\|w\|_{\lambda, \phi}^2 = \int_0^T \int_0^1 (\lambda^7 \phi^7 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda^3 \phi^3 |w_{xx}|^2 + \lambda \phi |w_{xxx}|^2) dx dt.$$

We first require the following

Lemma 3.2 *Under the hypothesis of Theorem 3.1, there exists $\delta > 0$ such that*

$$\langle P_1w, P_2w \rangle_{L^2(Q)} \geq \delta \|w\|_{\lambda, \phi}^2 + I_x \quad (30)$$

for λ large enough and for all $w \in \mathcal{W}_\lambda$.

Proof. It is sufficient to prove that

$$\langle P_1 w, P_2 w \rangle_{L^2} = \sum_{k=0}^3 I(w_{kx}) + R_0(w) + I_x \quad (31)$$

for a large enough λ , for all $w \in \mathcal{W}_\lambda$, where $|R_0(w)| \leq \lambda^{-1} \|w\|_{\lambda, \phi}^2$.

Let us first assume that we have (31). From hypotheses (20) to (22) we know that there exists $\varepsilon > 0$ such that ϕ satisfies for all $x \in (0, 1)$,

$$\begin{aligned} \phi_{xx}(x) &\leq -\varepsilon\phi < 0, \\ 30\phi_{xx}(x)\sigma(x) + 12\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0, \\ 58\phi_{xx}(x)\sigma(x) + 40\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0, \text{ and} \\ 2\phi_{xx}(x)\sigma(x) - 4\phi_x(x)\sigma_x(x) &\leq -\varepsilon\phi < 0. \end{aligned} \quad (32)$$

Furthermore, from (20) we can prove that $\phi \leq C\phi_x$. Then from (31) we obtain, for a large enough λ ,

$$\begin{aligned} \langle P_1 w, P_2 w \rangle_{L^2} &= \sum_{k=0}^3 I(w_{kx}) + R_0(w) + I_x \\ &\geq 2\delta \|w\|_{\lambda, \phi}^2 - |R_0(w)| + I_x \\ &\geq \delta \|w\|_{\lambda, \phi}^2 + I_x. \end{aligned} \quad (33)$$

Let us now prove (31): we write $\langle P_1 w, P_2 w \rangle_{L^2(Q)} = \sum_{i,j=1}^4 I_{i,j}$ where $I_{i,j}$ denotes the L^2 -product between the i -th term of $P_1 w$ in (27) and the j -th term of $P_2 w$ in (28).

Integrations by parts in time or space are performed on each expression $I_{i,j}$. Each resulting expression will be included in one of the terms on the right-hand side of (31). The results are listed below, and we indicate for each term where it will be included.

- $I_{1,1} = -I_{4,1} + \underbrace{3\lambda^2 \iint_Q (\phi_x^2 \sigma)_t |w_x|^2}_{R_0(w)}$
- $I_{1,2} = -12\lambda^5 \underbrace{\iint_Q (\phi_x^5 \sigma^2)_x |w_x|^2}_{I(w_x)}$
- $I_{1,3} = -12\lambda^3 \underbrace{\iint_Q (\phi_x^3 \sigma^2)_x |w_{xx}|^2}_{I(w_{2x})} + 12\lambda^3 \underbrace{\int_0^T \left[\phi_x^3 \sigma^2 |w_{xx}|^2 \right]_0^1}_{I_x}$
- $I_{1,4} = 12\lambda^5 \underbrace{\iint_Q [\phi_x^3 \sigma (\phi_x^2 \sigma)_x]_x |w|^2}_{R_0(w)} - 24\lambda^5 \underbrace{\iint_Q \phi_x^3 \sigma (\phi_x^2 \sigma)_x |w_x|^2}_{I(w_x)}$
- $I_{2,1} = -\frac{\lambda^4}{2} \underbrace{\iint_Q (\phi_x^4 \sigma)_t |w|^2}_{R_0(w)}$
- $I_{2,2} = -2\lambda^7 \underbrace{\iint_Q (\phi_x^7 \sigma^2)_x |w|^2}_{I(w)}$

- $I_{2,3} = \underbrace{-2\lambda^5 \iint_Q (\phi_x^5 \sigma^2)_{xxx} |w|^2}_{R_0(w)} + \underbrace{6\lambda^5 \iint_Q (\phi_x^5 \sigma^2)_x |w_x|^2}_{I(w_x)}.$
- $I_{2,4} = \underbrace{4\lambda^7 \iint_Q \phi_x^5 \sigma (\phi_x^2 \sigma)_x |w|^2}_{I(w)}.$
- $I_{3,1} = \frac{1}{2} \int_0^1 \left[\sigma |w_{xx}|^2 \right]_0^T = 0.$
- $I_{3,2} = \underbrace{-2\lambda^3 \iint_Q [(\phi_x^3 \sigma)_{xx} \sigma]_x |w_x|^2}_{R_0(w)} + \underbrace{4\lambda^3 \iint_Q (\phi_x^3 \sigma)_x \sigma |w_{xx}|^2}_{I(w_{2x})}$
 $+ \underbrace{2\lambda^3 \iint_Q (\phi_x^3)_x \sigma^2 |w_{xx}|^2}_{I(w_{2x})} - \underbrace{2\lambda^3 \int_0^T \left[\phi_x^3 \sigma^2 |w_{xx}|^2 \right]_0^1}_{I_x}.$
- $I_{3,3} = \underbrace{2\lambda \int_0^T \left[\phi_x \sigma \sigma_{xx} |w_{xx}|^2 \right]_0^1}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x \sigma \sigma_{xx})_x |w_{xx}|^2}_{R_0(w)} + \underbrace{8\lambda \iint_Q \phi_x \sigma \sigma_x |w_{3x}|^2}_{I(w_{xxx})}$
 $+ \underbrace{2\lambda \int_0^T \left[\phi_x \sigma^2 |w_{xx}|^2 \right]_0^1}_{I_x} - \underbrace{2\lambda \iint_Q (\phi_x \sigma^2)_x |w_{xxx}|^2}_{I(w_{xxx})}.$
- $I_{3,4} = \underbrace{4\lambda^3 \iint_Q (\phi_x (\phi_x^2 \sigma)_x)_{xx} \sigma w w_{xx}}_{R_0(w)} - \underbrace{4\lambda^3 \iint_Q (\phi_x (\phi_x^2 \sigma)_x)_x \sigma |w_x|^2}_{R_0(w)}$
 $+ \underbrace{4\lambda^3 \iint_Q \phi_x (\phi_x^2 \sigma)_x \sigma |w_{2x}|^2}_{I(w_{2x})}.$
- $I_{4,1} = 6\lambda^2 \iint_Q (\phi_x^2 \sigma)_x w_x w_t$, which is canceled when adding with $I_{1,1}$.
- $I_{4,2} = \underbrace{24\lambda^5 \iint_Q (\phi_x^2 \sigma)_x \phi_x^3 \sigma |w_x|^2}_{I(w_x)}.$
- $I_{4,3} = \underbrace{12\lambda^3 \iint_Q [(\phi_x^2 \sigma)_x \phi_x \sigma]_{xx} |w_x|^2}_{R_0(w)} - \underbrace{24\lambda^3 \iint_Q (\phi_x^2 \sigma)_x \phi_x \sigma |w_{xx}|^2}_{I(w_{2x})}.$
- $I_{4,4} = \underbrace{-12\lambda^5 \iint_Q (\phi_x^2 \sigma)_x (\phi_x^3 \sigma)_x |w|^2}_{R_0(w)}.$

Adding all the terms, we obtain (31). \square

Then, we will prove a Carleman inequality for the conjugated operator P .

Lemma 3.3 *There exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$ we have, for all $w \in \mathcal{W}_\lambda$,*

$$\int_0^T \int_0^1 (\lambda^7 \phi^7 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda^3 \phi^3 |w_{xx}|^2 + \lambda \phi |w_{xxx}|^2) dx dt$$

$$+ \|P_1 w\|_{L^2(Q)}^2 + \|P_2 w\|_{L^2(Q)}^2 \leq C \|P w\|_{L^2(Q)}^2 - I_x.$$

Proof.

From hypothesis (20) and inequalities listed in (32), we know that there exists $\delta > 0$ such that

$$\sum_{k=0}^3 I(w_{kx}) \geq \delta \|w\|_{\lambda, \phi}^2 \quad (34)$$

for a large enough parameter λ .

Besides, due to the definition of Rw in (29) and $\|q_i\|_{L^\infty((0,T) \times (0,1))} \leq m$ for $i = 0, 1, 2$ it is trivial to check that

$$\begin{aligned} \|Rw\|_{L^2((0,T) \times (0,1))}^2 &\leq C \left(\lambda^6 \iint_Q \phi^6 |w|^2 + \lambda^2 \iint_Q \phi^2 |w_x|^2 + \lambda^2 \iint_Q \phi^2 |w_{xx}|^2 \right) \\ &\leq C \lambda^{-1} \|w\|_{\lambda, \phi}^2. \end{aligned} \quad (35)$$

Thus, for a large enough λ , we have

$$\begin{aligned} \|P_1 w\|_{L^2}^2 + 2 \langle P_1 w, P_2 w \rangle + \|P_2 w\|_{L^2}^2 &= \|Pw - Rw\|_{L^2}^2 \\ &\leq 2 \|Pw\|_{L^2}^2 + 2 \|Rw\|_{L^2}^2 \\ &\leq 2 \|Pw\|_{L^2}^2 + C \lambda^{-1} \|w\|_{\lambda, \phi}^2. \end{aligned} \quad (36)$$

From Lemma 3.2 and estimates (36) and (34), we conclude the proof of Lemma 3.3. \square

To complete the proof of Theorem 3.1, we have to deal with the norms for $P_1 w$ and $P_2 w$ appearing in Lemma 3.3. From the definition of $P_2 w$, and because (20) holds, we have

$$\frac{1}{\lambda \phi} |w_t|^2 \leq \frac{2}{\lambda \phi} |P_2 w|^2 + C (\lambda^5 \phi^5 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda \phi |w_{xxx}|^2)$$

and

$$\iint_Q \frac{1}{\lambda \phi} |w_t|^2 \leq C \iint_Q |P_2 w|^2 + C \|w\|_{\lambda, \phi}^2$$

for a large enough λ . A similar result is proven for $(\sigma w_{xx})_{xx}$ and $P_1 w$, and we then have

$$\iint_Q \frac{1}{\lambda \phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) \leq C \iint_Q (|P_1 w|^2 + |P_2 w|^2) + C \|w\|_{\lambda, \phi}^2. \quad (37)$$

From (37) and Lemma 3.3 we obtain

$$\begin{aligned} \iint_Q \frac{1}{\lambda \phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) + \lambda^7 \phi^7 |w|^2 + \lambda^5 \phi^5 |w_x|^2 + \lambda^3 \phi^3 |w_{xx}|^2 + \lambda \phi |w_{xxx}|^2 dx dt \\ \leq C \iint_Q |Pw|^2 dx dt - C I_x. \end{aligned} \quad (38)$$

To handle the terms in I_x , we note that for any $x \in (0, 1)$ and a large enough λ ,

$$-C \lambda \int_0^T \phi_x(x, t) \sigma(x) \sigma_{xx}(x) |w_{xx}(x, t)|^2 dt \leq C \lambda^3 \int_0^T \phi_x(x, t)^3 \sigma(x)^2 |w_{xx}(x, t)|^2 dt.$$

Then

$$-C I_x \leq C \lambda^3 \int_0^T \phi_x(0, t)^3 \sigma(0)^2 |w_{xx}(0, t)|^2 dt + C \lambda \int_0^T \phi_x(0, t) \sigma(0)^2 |w_{xxx}(0, t)|^2 dt \quad (39)$$

and from (38) and (39) we obtain

$$\begin{aligned} \iint_Q \frac{1}{\lambda \phi} (|w_t|^2 + |(\sigma w_{xx})_{xx}|^2) + \|w\|_{\lambda, \phi}^2 &\leq C \iint_Q |Pw|^2 \\ &+ C \lambda^3 \int_0^T \phi_x(0, t)^3 \sigma(0)^2 |w_{xx}(0, t)|^2 dt + C \lambda \int_0^T \phi_x(0, t) \sigma(0)^2 |w_{xxx}(0, t)|^2 dt. \end{aligned} \quad (40)$$

Computing the derivatives of $e^{\lambda\phi}w$ it is trivial to prove that

$$\left| \partial_x^k v \right|^2 = \left| \partial_x^k (e^{\lambda\phi}w) \right|^2 \leq C \sum_{j=0}^k \left| \lambda^{k-j} \phi^{k-j} \partial_x^j w \right|^2$$

for each $k = 0, \dots, 3$. Therefore

$$\begin{aligned} \int_0^T \int_0^1 e^{-2\lambda\phi} \left(\lambda^7 \phi^7 |e^{\lambda\phi}w|^2 + \lambda^5 \phi^5 |(e^{\lambda\phi}w)_x|^2 + \lambda^3 \phi^3 |(e^{\lambda\phi}w)_{xx}|^2 + \lambda \phi |(e^{\lambda\phi}w)_{xxx}|^2 \right) dx dt \\ \leq C \|w\|_{\lambda, \phi}. \end{aligned}$$

Considering finally that $Pw = e^{-\lambda\varphi}Lv$, we obtain Carleman estimate (26). \square

Remark 3.4 *We considered the function β to be increasing. This allows the Carleman inequality to be obtained with boundary terms at $x = 0$. If a decreasing function β was used instead, then an inequality with boundary terms at $x = 1$ would have been obtained. As discussed in the following section, the boundary terms in the Carleman inequality are related to the location of the observations in the inverse problem.*

4 Inverse Problem

In this section, the local stability of the nonlinear inverse problem stated in Theorem 1.2 will be proved following the ideas of [6] and [13]. The proof is splitted in several steps.

Step 1. Local study of the inverse problem

Let γ , $\tilde{\gamma}$, y and \tilde{y} be defined as in Theorem 1.2. If we set $u = y - \tilde{y}$ and $f = \tilde{\gamma} - \gamma$, then u solves the following K-S equation:

$$\begin{cases} u_t + (\sigma(x)u_{xx})_{xx} + \gamma u_{xx} + \tilde{y}u_x + \tilde{y}_x u + uu_x = f(x)\tilde{y}_{xx}(x, t), & \forall (t, x) \in Q, \\ u(t, 0) = u(t, 1) = 0, & \forall t \in (0, T), \\ u_x(t, 0) = u_x(t, 1) = 0, & \forall t \in (0, T), \\ u(0, x) = 0, & \forall x \in (0, 1). \end{cases} \quad (41)$$

Then, in order to prove the stability of the inverse problem mentioned in the introduction, it is sufficient to obtain an estimate of f in terms of $u_{xx}(\cdot, 0)$, $u_{xxx}(\cdot, 0)$ and $u(T_0, \cdot)$, where $\tilde{\gamma}$ and \tilde{y} are given, $\gamma \in \mathcal{U}$ and u is the solution of Equation (41).

We begin by deriving Equation (41) with respect to time. Thus, $v = u_t$ satisfies the following equation:

$$\begin{cases} v_t + (\sigma v_{xx})_{xx} + \gamma v_{xx} + \tilde{y}v_x + \tilde{y}_x v = f\tilde{y}_{xxt} - g, & \forall (t, x) \in Q, \\ v(t, 0) = v(t, 1) = 0, & \forall t \in (0, T), \\ v_x(t, 0) = v_x(t, 1) = 0, & \forall t \in (0, T), \\ v(0, x) = fR(x, 0), & \forall x \in (0, 1), \end{cases} \quad (42)$$

where $g(x, t) = u(x, t)y_{xt}(x, t) + u_x(x, t)y_t(x, t)$.

The proof relies on the use of the Carleman estimate given in Theorem 3.1. This result will be used twice: First, Equation (42) allows to estimate v in terms of f , \tilde{y}_{xx} and g ; Then, Equation (41) will be used to handle the terms u and u_x , which appear in the expression of the source term g . The details are given below.

Step 2. First use of the Carleman estimate

Similarly to the proof of the Carleman estimate, we set $w = e^{-\lambda\phi}v$. Then, we work on the term

$$I = 2 \int_0^1 \int_0^{T_0} w(t, x) w_t(t, x) dt dx.$$

On the one hand, we can calculate I and bound it from below. Indeed, using $w(0, x) = e^{-\lambda\phi(0, x)}v(0, x) = 0$ for all $x \in (0, 1)$ and Equation (41), we can easily obtain

$$\begin{aligned} I &= \int_0^1 |w(T_0, x)|^2 dx \\ &= \int_0^1 e^{-2\lambda\phi(T_0, x)} |(f\tilde{y}_{xx} - (\sigma u_{xx})_{xx} - \gamma u_{xx} - \tilde{y}u_x - \tilde{y}_x u - uu_x)(T_0, x)|^2 dx \\ &\geq \int_0^1 e^{-2\lambda\phi(T_0, x)} |f(x)|^2 |\tilde{y}_{xx}(T_0, x)|^2 dx - C \|u(T_0)\|_{H^4(0,1)}^2 - C \|u(T_0)\|_{H^1(0,1)}^4 \end{aligned}$$

where C depends on $\|\gamma\|_{L^\infty(0,1)}$, $\|\tilde{y}(T_0)\|_{W^{1,\infty}(0,1)}$ and $\|\sigma\|_{W^{2,\infty}(0,1)}$.

On the other hand, in order to estimate I from above we apply the Carleman estimate (40) to Equation (42) using $q_0 = \tilde{y}_x$ and $q_1 = \tilde{y}$, which are uniformly bounded in $L^\infty((0, T) \times (0, 1))$ by the hypothesis in Theorem 1.2. We obtain that

$$\begin{aligned} I &= 2 \int_0^1 \int_0^{T_0} w(t, x) w_t(t, x) dt dx \\ &\leq \left(\int_0^1 \int_0^{T_0} \lambda \phi(t, x) |w(t, x)|^2 dt dx \right)^{\frac{1}{2}} \left(\int_0^1 \int_0^{T_0} \frac{|w_t(t, x)|^2}{\lambda \phi(t, x)} dt dx \right)^{\frac{1}{2}} \\ &\leq C \lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |f(x) \tilde{y}_{xx}(x, t)|^2 dx dt + C \lambda^{-3} \int_0^1 \int_0^T e^{-2\lambda\phi} |g(x, t)|^2 dx dt \\ &\quad + C \lambda^{-3} \int_0^T e^{-2\lambda\phi(0, t)} (\lambda^3 \phi_x^3(0, t) \sigma^2(0) |v_{xx}(0, t)|^2 + \lambda \phi_x(0, t) \sigma^2(0) |v_{xxx}(0, t)|^2) dt. \end{aligned}$$

Step 3. Second use of the Carleman estimate

Considering that $g = uy_{xt} + u_x y_t$, we will now apply the Carleman estimate to Equation (41) in order to manage the term in g of the previous inequality. The unknown trajectory y is nevertheless such that y_{xt} and y_t belong to $L^\infty(0, T; L^\infty(0, 1))$ since $y \in H^1(0, T; H^4(0, 1))$. Thus, we have

$$\begin{aligned} \iint_Q e^{-2\lambda\phi} |g(x, t)|^2 dx dt &\leq 2 \iint_Q e^{-2\lambda\phi} |u|^2 |y_{xt}|^2 dx dt + 2 \iint_Q e^{-2\lambda\phi} |u_x|^2 |y_t|^2 dx dt \\ &\leq C \iint_Q e^{-2\lambda\phi} (|u|^2 + |u_x|^2) dx dt. \end{aligned}$$

The Carleman estimate (40) is applied to equation (41), using the identity $\tilde{y}_x u + uu_x = uy_x$, and taking $q_0 = y_x$ and $q_1 = \tilde{y}$, which are bounded in $L^\infty((0, T) \times (0, 1))$.

Therefore, we can choose λ_0 as large as possible in Theorem 3.1: we then obtain

$$\begin{aligned}
& \iint_Q e^{-2\lambda\phi} |g(x,t)|^2 \leq C\lambda^{-5} \iint_Q e^{-2\lambda\phi} (\lambda^7 |u|^2 + \lambda^5 |u_x|^2) \\
& \leq C\lambda^{-5} \iint_Q e^{-2\lambda\phi} |f(x)\tilde{y}_{xx}(x,t)|^2 \\
& \quad + C\lambda^{-5} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt.
\end{aligned}$$

Gathering all of the estimates of I and g that were obtained above, we have

$$\begin{aligned}
& \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 |\tilde{y}_{xx}(T_0,x)|^2 dx - C \|u(T_0)\|_{H^4(0,1)}^2 - C \|u(T_0)\|_{H^1(0,1)}^4 \\
& \leq C\lambda^{-3} \iint_Q e^{-2\lambda\phi} |f(x)\tilde{y}_{xxt}(x,t)|^2 dx dt + C\lambda^{-8} \iint_Q e^{-2\lambda\phi} |f(x)\tilde{y}_{xx}(x,t)|^2 dx dt \\
& \quad + C\lambda^{-8} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt \\
& \quad + C\lambda^{-3} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |v_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |v_{xxx}(0,t)|^2) dt.
\end{aligned}$$

From the hypothesis of the theorem, we have $\tilde{y}_{xx} \in L^\infty(0,T; W^{1,\infty}(0,1))$, $|\tilde{y}_{xx}(T_0, \cdot)| > r > 0$ in $(0,1)$. Using also that the Carleman weight function satisfies $e^{-2\lambda\phi(t,x)} \leq e^{-2\lambda\phi(T_0,x)}$ in $(0,T) \times (0,1)$, we obtain

$$\begin{aligned}
& \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 dx \\
& \leq C \left(\lambda^{-3} \int_0^1 e^{-2\lambda\phi(T_0,x)} |f(x)|^2 dx + \|u(T_0)\|_{H^4(0,1)}^2 + \|u(T_0)\|_{H^1(0,1)}^4 \right. \\
& \quad + \lambda^{-8} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |u_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |u_{xxx}(0,t)|^2) dt \\
& \quad \left. + \lambda^{-3} \int_0^T e^{-2\lambda\phi(0,t)} (\lambda^3 \phi_x^3(0,t) \sigma^2(0) |v_{xx}(0,t)|^2 + \lambda \phi_x(0,t) \sigma^2(0) |v_{xxx}(0,t)|^2) dt \right).
\end{aligned}$$

Therefore, by choosing λ large enough, we prove that there exists a constant C that depends on r, K, T, λ_0, m such that

$$\begin{aligned}
\|f(x)\|_{L^2(0,1)}^2 & \leq C \left(\|u(T_0, x)\|_{H^4(0,1)}^2 + \|u(T_0, x)\|_{H^1(0,1)}^4 \right. \\
& \quad \left. + \|u_{xx}(t, 0)\|_{H^1(0,T)}^2 + \|u_{xxx}(t, 0)\|_{H^1(0,T)}^2 \right).
\end{aligned}$$

This estimate leads to the stability of the initial inverse problem and we have obtained the first estimate in Theorem 1.2.

Step 4. Regularity estimate

We will now show that the second estimate also holds. Indeed, using $T_0 \in (0, T)$ and using that $H^1(0, T; H^p(0, 1)) \hookrightarrow C([0, T]; H^p(0, 1))$, we have both, for some constant C ,

$$\begin{aligned}
\|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^4(0,1)} & \leq C \|y - \tilde{y}\|_{H^1(0,T; H^4(0,1))}, \\
\|y(T_0, \cdot) - \tilde{y}(T_0, \cdot)\|_{H^1(0,1)} & \leq C \|y - \tilde{y}\|_{H^1(0,T; H^1(0,1))}.
\end{aligned}$$

Since $(y - \tilde{y}) \in L^2(0, T; H^4 \cap H_0^2(0, 1))$, we have $(y_{xxx} - \tilde{y}_{xxx}) \in L^2(0, T; H^1(0, 1))$ and as $L^2(0, T; H^1(0, 1)) \hookrightarrow L^2((0, T); C([0, 1]))$, we get the estimates

$$\begin{aligned} \|y_{xx}(\cdot, 0) - \tilde{y}_{xx}(\cdot, 0)\|_{L^2(0, T)} &\leq C \|y - \tilde{y}\|_{L^2(0, T; H^4(0, 1))}, \\ \|y_{xxx}(\cdot, 0) - \tilde{y}_{xxx}(\cdot, 0)\|_{L^2(0, T)} &\leq C \|y - \tilde{y}\|_{L^2(0, T; H^4(0, 1))}. \end{aligned}$$

Using the same, but starting with $(y_t - \tilde{y}_t) \in L^2(0, T; H^4 \cap H_0^2(0, 1))$, we obtain the estimates

$$\begin{aligned} \|y_{txx}(\cdot, 0) - \tilde{y}_{txx}(\cdot, 0)\|_{L^2(0, T)} &\leq C \|y_t - \tilde{y}_t\|_{L^2(0, T; H^4(0, 1))}, \\ \|y_{txxx}(\cdot, 0) - \tilde{y}_{txxx}(\cdot, 0)\|_{L^2(0, T)} &\leq C \|y_t - \tilde{y}_t\|_{L^2(0, T; H^4(0, 1))}. \end{aligned}$$

These inequalities provide the second estimate in (6) and conclude the proof of Theorem (1.2).

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